NORMANHURST BOYS HIGH SCHOOL

# MATHEMATICS EXTENSION 1

2019 Year 12 Course Trial Examination
Friday, 16 August 2019

**General instructions**

- Working time – 2 hours.
  (plus 5 minutes reading time)
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- NESA approved calculators may be used.
- Attempt all questions.
- At the conclusion of the examination, bundle the booklets used in the correct order behind this paper and hand to examination supervisors.

**SECTION I**

- Mark your answers on the answer grid provided

**SECTION II**

- Commence each new question on a new booklet. Write on both sides of the paper.
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.

**NESA STUDENT #:** ........................................................ **# BOOKLETS USED:** ................................

Class: (please ✓)

○ 12MAT.1 – Mr Lam
○ 12MAT.2 – Mrs Bhamra
○ 12MAT.3 – Mrs Gan
○ 12MAT.4 – Ms Park
○ 12MAT.5 – Mr Sekaran

Marker's use only

| QUESTION | 1-10 | 11 | 12 | 13 | 14 | Total | % |
|---|---|---|---|---|---|---|---|
| MARKS | $\overline{10}$ | $\overline{15}$ | $\overline{15}$ | $\overline{15}$ | $\overline{15}$ | $\overline{70}$ | |

# Section I

**10 marks**
**Attempt Questions 1 to 10**
**Allow approximately 10 minutes for this section**

Mark your answers on the answer grid provided.

**Questions** **Marks**

**1.** Which expression is equal to $\int \sin^2 2x\, dx$? **1**

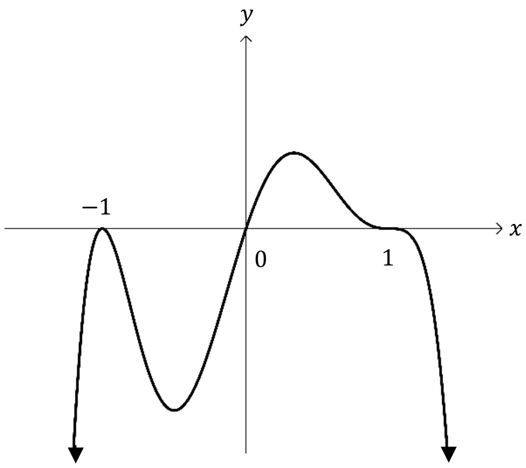
(A) $\frac{1}{8}(4x + \sin 4x) + c$

(B) $\frac{1}{8}(4x - \sin 4x) + c$

(C) $-\frac{\cos^3 2x}{6} + c$

(D) $\frac{\sin^3 2x}{6} + c$

**2.** Which of the following could be the equation of the graph given below? **1**




(A) $y = -\frac{x}{2}(1-x)^3(1-x)^2$

(B) $y = -\frac{x}{2}(1-x)^2(x+1)^3$

(C) $y = 3x(1-x)^3(x+1)^2$

(D) $y = 4x(1-x)^2(x-1)^3$

**3.** $x = 0.5$ is the first approximation to the positive root of the function $f(x) = xe^x - 1$. **1**

Which of the following is the second approximation using Newton's method, correct to two decimal places?

(A) $x = 0.56$

(B) $x = 0.57$

(C) $x = 0.58$

(D) $x = 0.59$

**4.** Which of the following is equal to $\lim_{x \to 0} \left( \frac{\sin 4x}{2x \cos 2x} \right)$? **1**

(A) $\frac{1}{2}$

(B) $0$

(C) $4$

(D) $2$

**5.** The point $P$ divides the interval from $A(2, -5)$ to $B(-4,1)$, internally in the ratio $1:2$. **1**

What are the coordinates of $P$?

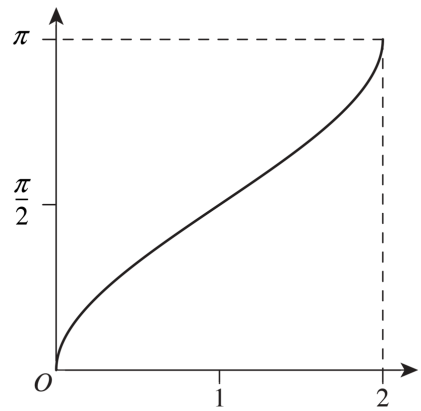
(A) $(-\frac{14}{3}, \frac{5}{3})$

(B) $(-\frac{14}{3}, -3)$

(C) $(0, -3)$

(D) $(-8,11)$

**6.** Which function best describes the graph below? **1**




(A) $y = \cos^{-1} x$

(B) $y = 1 - \cos^{-1} x$

(C) $y = \cos^{-1}(x - 1)$

(D) $y = \cos^{-1}(1 - x)$

**7.** Which of the following is equal to $\frac{d}{dx}(\cos^{-1}\frac{x}{4})$ ? **1**

(A) $-\frac{1}{\sqrt{4-x^2}}$

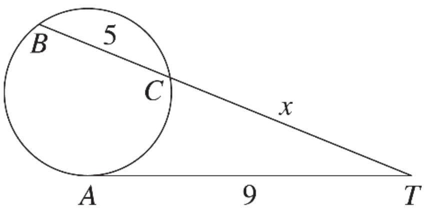
(B) $-\frac{1}{2\sqrt{4-x}}$

(C) $-\frac{4}{\sqrt{16-x^2}}$

(D) $-\frac{1}{\sqrt{16-x^2}}$

**8.** The line $AT$ is the tangent to the circle at $A$ and the line $BT$ is a secant meeting the circle at $B$ and $C$. **1**
$AT = 9$, $BC = 5$ and $CT = x$.

Which of the following equations is correct?




(A) $x^2 + 5x - 81 = 0$

(B) $x^2 + 5x + 81 = 0$

(C) $x^2 - 5x - 81 = 0$

(D) $x^2 + 5x - 9 = 0$

**9.** If $y = \cos(\ln x)$, which is equal to $\frac{d^2y}{dx^2}$ ? **1**

(A) $\frac{\cos(\ln x)+\sin(\ln x)}{x^2}$

(B) $\frac{x^2\cos(\ln x)+\sin(\ln x)}{x^2}$

(C) $\frac{\sin(\ln x)-x^2\cos(\ln x)}{x^2}$

(D) $\frac{\sin(\ln x)-\cos(\ln x)}{x^2}$

**10.** Which inequality has the same solution as $|x+3| + |x-4| = 7$? **1**

(A) $|2x-1| \geq 7$

(B) $x^2 - x - 12 \leq 0$

(C) $\frac{7}{3-x} \geq 1$

(D) $\frac{1}{x-4} - \frac{1}{x+3} \leq 0$

# Section II

**60 marks**
**Attempt Questions 11 to 14**
**Allow approximately 1 hour and 50 minutes for this section.**

Write your answers in the writing booklets supplied. Additional writing booklets are available.
Your responses should include relevant mathematical reasoning and/or calculations.

**Question 11** (15 marks) **Marks**

(a) Find all the asymptotes of $y = \frac{5x}{x^2+x-12}$. **2**

(b) Solve $\frac{1}{2x} \geq -\frac{1}{x-2}$. **4**

(c) The roots of $x^2 - 2x - 1 = 0$ are $\tan\alpha$ and $\tan\beta$. **3**
Given that $\alpha$ and $\beta$ are acute, find the exact value of $\alpha + \beta$.

(d) Find the size of the acute angle between the curves $f(x) = (x+2)^2 + 4$ and $g(x) = x^2 - 4$ at their point of intersection. Give your answer to the nearest degree. **3**

(e) The region enclosed by the graph of $y = \frac{1}{\sqrt{1+9x^2}}$, the line $x = 0$, the line $x = \frac{1}{3}$ and the $x$-axis is rotated about the $x$-axis to form a solid of revolution. **3**

Find the exact volume of the solid formed.

**Question 12** (15 marks) Use a SEPARATE Writing Booklet.

(a) A student tosses two regular dice with faces numbered 1 to 6. He records the maximum of the two uppermost faces as a score.

i. Find the probability that he records the score 1 in a single throw of the two dice. **1**

ii. Find the probability that he records the score 6 in a single throw of the two dice. **1**

(b) Consider the polynomial $P(x) = x^3 + bx^2 + cx - 18$. **3**

Given that $(x + 2)$ is a factor of $P(x)$ and $-8$ is the remainder when $P(x)$ is divided by $(x + 1)$, find the values of $b$ and $c$.

(c) Evaluate, by using the substitution $u^2 = 1 + x$: **3**

$$\int_0^3 \frac{x(x+2)}{\sqrt{1+x}}\,dx$$

(d) i. Express $\cos x - \sqrt{3}\sin x$ in the form $R\cos(x + \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. **2**

iii. Hence or otherwise, find the general solution of the equation $\cos x - \sqrt{3}\sin x = \sqrt{2}$. **2**

(e) The rate at which a body cools in air is assumed to be proportional to the difference between its temperature $T$ and the constant room temperature $P$ of the surrounding air.
This can be expressed as:

$$T = P + Ae^{kt}$$

where $t$ is time in hours and $k$ is a constant.

A cup of tea cools from 85°C to 25°C in 90 minutes. The air temperature $P$ in the room is 20°C.

i. Show that $k = \frac{2}{3}\log_e \frac{1}{13}$. **2**

ii. Find the temperature of the cup of tea after a further 30 minutes. **1**
Give your answer correct to 2 decimal places.

**Question 13** (15 marks) Use a SEPARATE Writing Booklet.

(a) By letting $t = \tan\theta$, prove that: **2**

$$\frac{\tan 2\theta + \cot\theta}{\tan 2\theta - \tan\theta} = \cot^2 x$$

(b) The velocity, $v$ ms$^{-1}$, of a particle moving along a straight line is shown in the graph below.

Initially, the particle is at rest at the origin and returns to the origin at $t = 7$.

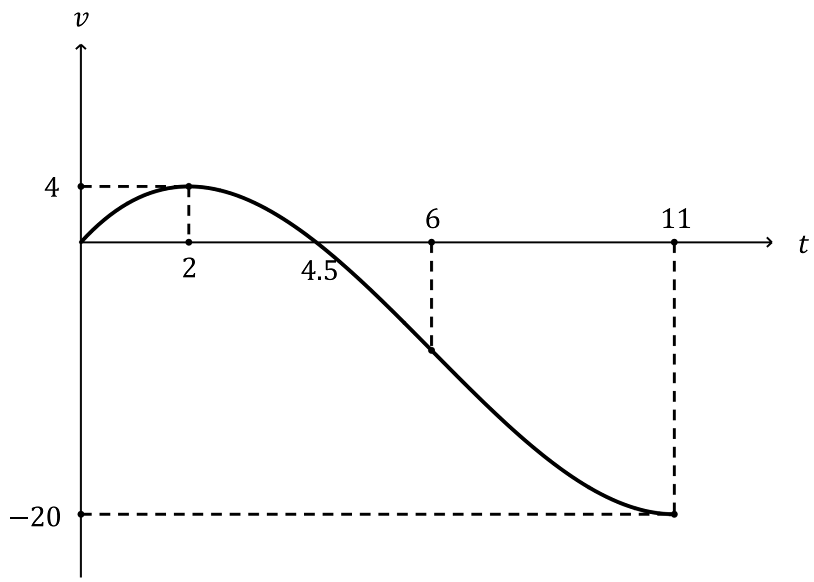



*Diagram not drawn to scale.*

i. When is the speed of the particle the greatest? **1**

ii. How many times does the particle change direction? **1**

iii. When is the acceleration of the particle the greatest after it starts moving? **1**

iv. Sketch the graph of the displacement function $x(t)$ for $0 \le t \le 7$, showing all important features. **2**

(c) Consider the function $f(x) = \sqrt{2x-1} + 1$.

i. Find the equation of the inverse function $f^{-1}(x)$, stating the domain. **2**

ii. Find the $x$-coordinates of the points where the graphs $y = f(x)$ and $y = f^{-1}(x)$ intersect. **2**

(d) Liquid flows into a container made in the shape of a hollow, inverted right circular cone with radius 16 cm and height 24 cm, as shown in the diagram.

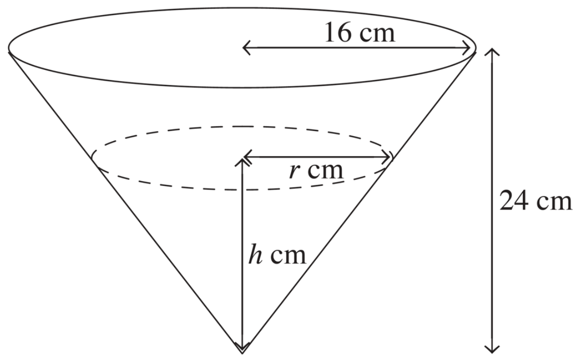



*Diagram not drawn to scale.*

At time $t$ seconds, the height of the liquid is $h$ cm, the surface of the liquid has radius $r$ cm and the volume of the liquid is $V$ cm$^3$.

i. Show that the $V = \frac{4\pi h^3}{27}$. **2**

ii. Liquid flows into the container at a rate of 8 cm$^3$s$^{-1}$. **2**

Show that $\frac{dh}{dt} = \frac{1}{2\pi}$ when $h = 6$.

**Question 14** (15 marks) Use a SEPARATE Writing Booklet.

(a) Use mathematical induction to prove that, for every positive integer $n$: **3**

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{1}{4}n^2(n+1)^2$$

(b) Consider the following acute-angled triangle $ABC$.

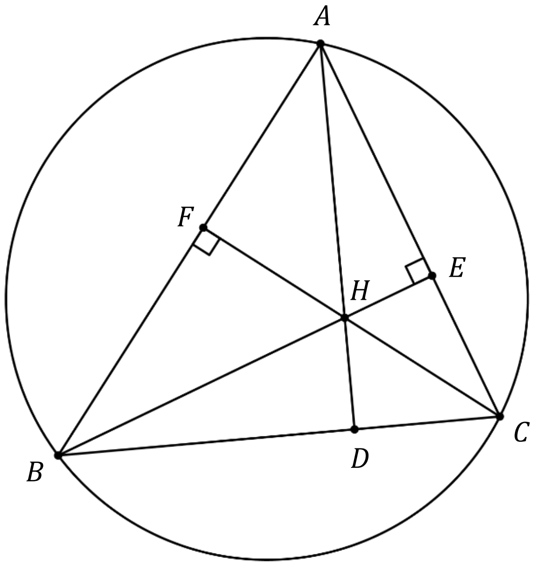



*Diagram not drawn to scale.*

Points $E$ and $F$ lie on $AC$ and $AB$ respectively, such that $BE$ is perpendicular to $AC$ and $CF$ is perpendicular to $AB$. The line segments $BE$ and $CF$ meet at $H$.
The line segment $AH$ produced meets $BC$ at $D$.

Copy or trace the diagram into your writing booklet.

i. Give a reason for why $CEFB$ is a cyclic quadrilateral. **1**

ii. By considering cyclic quadrilaterals, show that $\angle CAD = \angle EFH = \angle EBC$, giving full reasons. **3**

iii. Hence, prove that $AD$ is perpendicular to $BC$. **2**

(c) The points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola with equation $x^2 = 4ay$, where $p \neq 0, q \neq 0$ and $p \neq q$. The parabola has a focus $S(0, a)$.

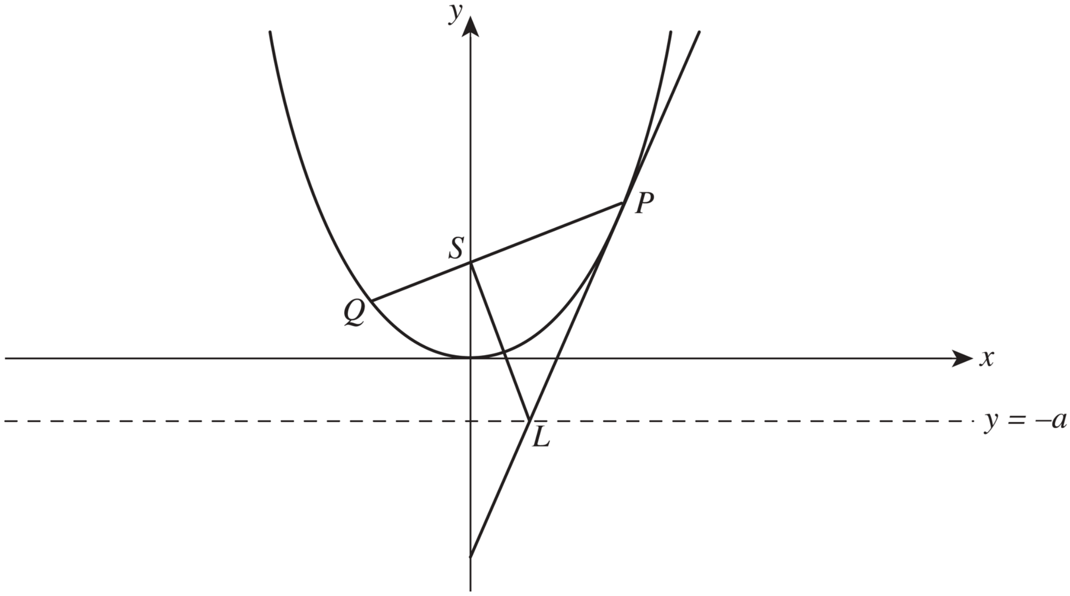



*Diagram not drawn to scale.*

Let $L$ be the intersection of the tangent at $P$ and the directrix, and $PQ$ is a focal chord.

i. Show that $pq = -1$. **1**

ii. Explain why $PS = a(p^2 + 1)$. **1**

iii. Show that $PS \times QS = LS^2$. **4**

**End of examination.**

Q1) $\int \sin^2 2x \, dx$

$= \int \frac{1}{2}(1-\cos 4x)\, dx$

$= \frac{1}{2}\left(x - \frac{1}{4}\sin 4x\right) + c$

$= \frac{1}{8}(4x - \sin 4x) + c$ B

Q2) C

Q3) $f(x) = xe^x - 1$

$f'(x) = xe^x + e^x$

$= e^x(x+1)$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 0.5 - \frac{f(0.5)}{f'(0.5)}$$

$$= 0.5 - \frac{0.5e^{0.5} - 1}{e^{0.5}(0.5+1)}$$

$= 0.5710 \ldots$

$\doteqdot 0.57$ B

Q4) $$\frac{\sin 4x}{2x\cos 2x} = \frac{\cancel{2}\sin 2x \cancel{\cos 2x}}{\cancel{2}x\cancel{\cos 2x}}$$

$$= \frac{\sin 2x}{x}$$

$$= 2 \cdot \frac{\sin 2x}{2x}$$

$$\therefore \lim_{x\to 0} \frac{\sin 4x}{2x\cos 2x} = 2\lim_{x \to 0} \frac{\sin 2x}{2x}$$

$= 2 \cdot 1$

$= 2$ D

Q5) $A(2,-5)$ $B(-4,1)$

$1 : 2$

$$P = \left(\frac{1\times -4 + 2\times 2}{1+2}, \frac{1\times 1 + 2\times -5}{1+2}\right)$$

$= (0, -3)$ C

Q6) The graph shows $y=\cos^{-1}x$ reflected in the y-axis shifted right by 1. D

Q7) $\frac{d}{dx}\cos^{-1}\frac{1}{4}x$

$$= -\frac{1}{\sqrt{1-\left(\frac{x}{4}\right)^2}}\cdot\frac{1}{4}$$

$$= -\frac{1}{4\sqrt{\frac{16-x^2}{16}}}$$

$$= -\frac{1}{\sqrt{16-x^2}}$$ D

Q8) $AT^2 = CT \times BT$

$9^2 = x\cdot(x+5)$

$0 = x^2+5x-81$ A

Q9) $y=\cos(\ln x)$

$y' = -\frac{1}{x}\sin(\ln x)$

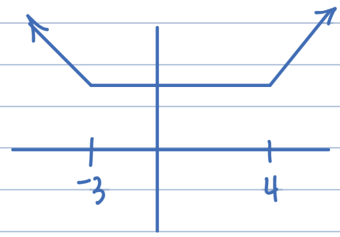
$= -x^{-1}\sin(\ln x)$

$y'' = 1x^{-2}\sin(\ln x) + -x^{-1}\cdot\frac{1}{x}\cos(\ln x)$

$$= \frac{\sin(\ln x)-\cos(\ln x)}{x^2}$$

D

Q10) $|x+3|+|x-4| = \begin{cases} -2x+1, & x<-3 \\ 7, & -3\le x\le 4 \\ 2x-1, & x>4 \end{cases}$




$x^2-x-12\le 0$

$(x+3)(x-4)\le 0$

$-3\le x\le 4$ B

Q11) a) $y = \dfrac{5x}{x^2+x-12}$

$= \dfrac{5x}{(x-3)(x+4)}$

asymptotes: $x = 3, -4$ ✓

$y = 0$ ✓

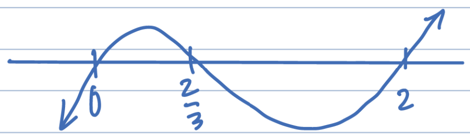
b) $\dfrac{1}{2x} \geq -\dfrac{1}{x-2}$ , $x \neq 0, 2$

$x(x-2)^2 \geq -2x^2(x-2)$ ✓ multiply by $2x^2(x-2)^2$

$x(x-2)^2 + 2x^2(x-2) \geq 0$

$x(x-2)[x-2+2x] \geq 0$

$x(x-2)(3x-2) \geq 0$ ✓




$\therefore\ 0 < x \leq \frac{2}{3}$ , $x > 2$ ✓

✓(excl. $x = 0, 2$)

c) $x^2 - 2x - 1 = 0$

$\tan\alpha + \tan\beta = \dfrac{-(-2)}{1}$

$= 2$

$\tan\alpha . \tan\beta = \dfrac{-1}{1}$

$= -1$ ✓

$\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

$= \dfrac{2}{1-(-1)}$

$= 1$ ✓

$\alpha + \beta = \frac{\pi}{4}$ ✓

d) $f(x) = g(x)$

$(x+2)^2 + 4 = x^2 - 4$

$\cancel{x^2} + 4x + 4 + 4 = \cancel{x^2} - 4$

$4x = -12$

$x = -3$ ✓

$f'(x) = 2(x+2)$

$m_1 = 2(-3+2)$

$= -2$

$g'(x) = 2x$

$m_2 = 2(-3)$

$= -6$

✓

$\tan\theta = \left| \frac{-2 - -6}{1 + -2(-6)} \right|$

$= \frac{4}{13}$

$\theta = \tan^{-1}\left(\frac{4}{13}\right)$

$= 17.10\ldots$

$\doteqdot 17°$ ✓

e) $V = \pi \int y^2 \, dx$

$= \pi \int_0^{\frac{1}{3}} \frac{1}{1+9x^2} \, dx$ ✓

$= \pi \int_0^{\frac{1}{3}} \frac{1}{1+(3x)^2} \, dx$

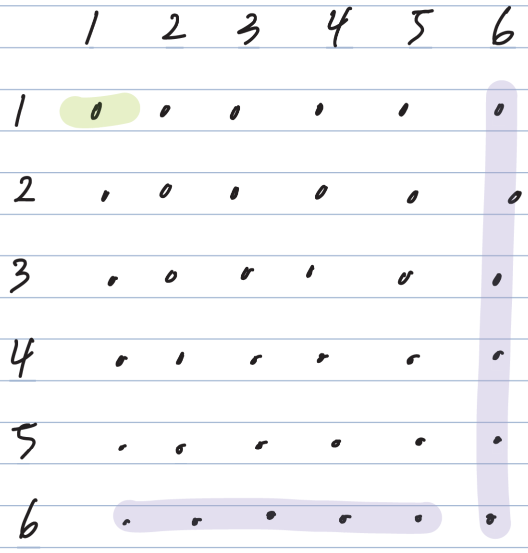
$= \pi \left[ \frac{1}{3} \tan^{-1}(3x) \right]_0^{\frac{1}{3}}$ ✓

$= \frac{\pi}{3} \left( \tan^{-1}\left(3 \cdot \frac{1}{3}\right) - \tan^{-1}(0) \right)$

$= \frac{\pi}{3} \left( \frac{\pi}{4} - 0 \right)$

$= \frac{\pi^2}{12}$ units$^3$ ✓

Q(2) a)



i) $P(1,1) = \frac{1}{36}$ ✓

ii) $P(6,6) + P(6, \text{other}) + P(\text{other}, 6) = \frac{11}{36}$ ✓

b) $P(x) = x^3 + bx^2 + cx - 18$

$$P(-2) = (-2)^3 + b(-2)^2 + c(-2) - 18 = 0$$
$$-8 + 4b - 2c - 18 = 0$$
$$2b - c = 13 \quad —①$$
✓

$$P(-1) = (-1)^3 + b(-1)^2 + c(-1) - 18 = -8$$
$$-1 + b - c - 18 = -8$$
$$b - c = 11 \quad —②$$
✓

① − ② : $b = 2$

↳ $2 - c = 11$

$c = -9$

✓

c) $u^2 = 1 + x$

$$x = u^2 - 1$$
$$dx = 2u\,du$$
$$x = 3 \to u = 2$$
$$x = 0 \to u = 1$$

$$\int_0^3 \frac{x(x+2)}{\sqrt{1+x}}\,dx = \int_1^2 \frac{(u^2-1)(u^2+1)}{\cancel{\sqrt{u^2}}} 2\cancel{u}\,du$$

$$= 2\int_1^2 u^4 - 1\,du$$

$$= 2\left[\frac{u^5}{5} - u\right]_1^2$$

$$= 2\left(\frac{2^5}{5} - 2 - \left(\frac{1^5}{5} - 1\right)\right)$$

$$= 2 \cdot \frac{26}{5}$$

$$= \frac{52}{5}$$

d) i) $\cos x - \sqrt{3}\sin x \equiv R\cos(x+\alpha)$

$= R\cos x\cos\alpha - R\sin x\sin\alpha$

$$\left.\begin{aligned} 1 &= R\cos\alpha \\ \sqrt{3} &= R\sin\alpha \end{aligned}\right\} \quad \tan\alpha = \sqrt{3}$$

$\alpha = \frac{\pi}{3}$ ✓

$R^2 = 1+3$

$R = 2$ ✓

$\therefore \cos x - \sqrt{3}\sin x = 2\cos\left(x+\frac{\pi}{3}\right)$

ii) $\cos x - \sqrt{3}\sin x = \sqrt{2}$

$2\cos\left(x+\frac{\pi}{3}\right) = \sqrt{2}$

$\cos\left(x+\frac{\pi}{3}\right) = \frac{1}{\sqrt{2}}$ ✓

$x+\frac{\pi}{3} = 2\pi n \pm \frac{\pi}{4}, \quad n \in \mathbb{Z}$

$x = 2\pi n \pm \frac{\pi}{4} - \frac{\pi}{3}$ ✓

e) i) $T = P + Ae^{kt}$

$t=0 \rightarrow 85 = 20 + Ae^{0}$

$A = 65$ ✓

$$\left.\begin{aligned} t=1.5 \rightarrow 25 &= 20 + 65e^{k(1.5)} \\ \frac{1}{13} &= e^{1.5k} \\ \log_e \frac{1}{13} &= 1.5k \\ k &= \frac{2}{3}\log_e \frac{1}{13} \end{aligned}\right\} ✓$$

ii) $t=2 \rightarrow T = 20 + 65e^{k(2)}$

$= 22.1264\ldots$

$\doteqdot 22.13^\circ$ ✓

Q(3) a) $LHS = \dfrac{\tan 2\theta + \cot\theta}{\tan 2\theta - \tan\theta}$

$= \left(\dfrac{2t}{1-t^2} + \dfrac{1}{t}\right) \div \left(\dfrac{2t}{1-t^2} - t\right)$, $\tan\theta = t$

$= \dfrac{2t^2 + 1 - t^2}{t(1-t^2)} \div \dfrac{2t - t(1-t^2)}{1-t^2}$ ✓

$= \dfrac{t^2+1}{t\cancel{(1-t^2)}} \times \dfrac{\cancel{1-t^2}}{t+t^3}$

$= \dfrac{\cancel{t^2+1}}{t} \times \dfrac{1}{t\cancel{(1+t^2)}}$ ✓

$= \dfrac{1}{t^2}$

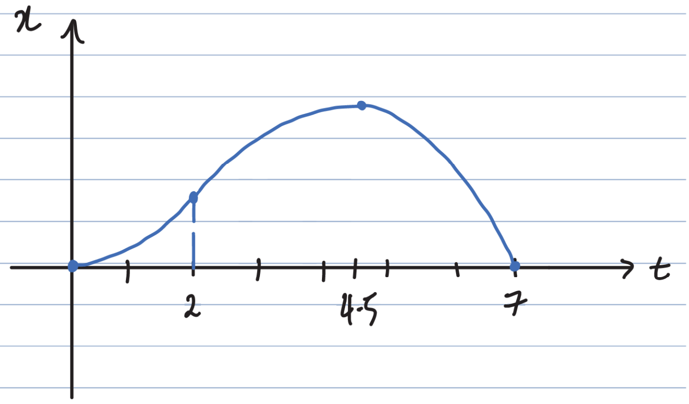
$= \cot^2\theta$

$= RHS$

b) i) $t = 11$ (20m/s)

ii) once (at $t = 4.5$)

iii) $t = 6$

iv) 


✓ shape

✓ turning pts $x = 0, 4.5$
and $x$-int $x = 0, 7$
pt of inflexion $x = 2$

c) i) $y = \sqrt{2x-1} + 1$

$x = \sqrt{2y-1} + 1$

$(x-1)^2 = 2y - 1$ ✓

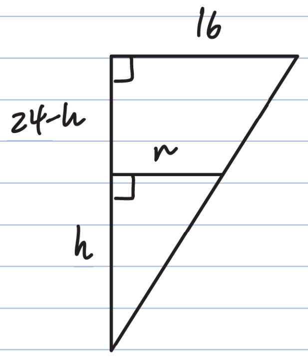
$y = \frac{1}{2}\left((x-1)^2 + 1\right)$, $x \geq 1$ ✓

ii) $\sqrt{2x-1}+1=x$

$$2x-1=(x-1)^2$$
$$=x^2-2x+1$$
$$0=x^2-4x+2 \checkmark$$

$$x=\frac{4\pm\sqrt{4^2-4(1)(2)}}{2(1)}$$
$$=\frac{4\pm\sqrt{8}}{2}$$
$$=2\pm\sqrt{2}$$
$$=2+\sqrt{2} \quad (x>1) \checkmark$$

d) i)




$$\frac{r}{h}=\frac{16}{24}$$
$$r=\frac{2h}{3} \checkmark$$

$$V=\frac{1}{3}\pi r^2 h$$
$$=\frac{1}{3}\pi\left(\frac{2h}{3}\right)^2\cdot h$$
$$=\frac{4\pi}{27}h^3 \checkmark$$

ii) $\frac{dV}{dt}=8$

$$\frac{dV}{dh}=\frac{4\pi}{9}h^2 \checkmark$$

$$\frac{dV}{dt}=\frac{dV}{dh}\cdot\frac{dh}{dt}$$
$$8=\frac{4\pi}{9}h^2\cdot\frac{dh}{dt}$$

$$\left.\begin{aligned}\frac{dh}{dt}&=\frac{18}{\pi h^2}\\ h=6 \rightarrow \frac{dh}{dt}&=\frac{18}{36\pi}\\ &=\frac{1}{2\pi}\end{aligned}\right\} \checkmark$$

Q(4) a) $1^3 + 2^3 + 3^3 + \dots + n^3 = \frac{1}{4}n^2(n+1)^2$

prove true for $n=1$.

$LHS = 1^3$

$RHS = \frac{1}{4}(1)^2(1+1)^2$

$= 1$

$= LHS$

$\therefore$ The result is true for $n=1$ ✓

Assume true for $n=k$.

ie. $1^3 + 2^3 + 3^3 + \dots + k^3 = \frac{1}{4}k^2(k+1)^2$

prove true for $n=k+1$.

ie. $1^3 + 2^3 + 3^3 + \dots + k^3 + (k+1)^3 = \frac{1}{4}(k+1)^2(k+2)^2$

$LHS = 1^3 + 2^3 + 3^3 + \dots + k^3 + (k+1)^3$

$= \frac{1}{4}k^2(k+1)^2 + (k+1)^3$

$= \frac{1}{4}(k+1)^2\left[k^2 + 4(k+1)\right]$

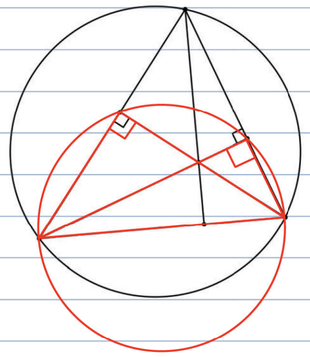
$= \frac{1}{4}(k+1)^2(k+2)^2$

$= RHS$

$\therefore$ The result is true for $n=k+1$ ✓

Hence the statement is true for all positive integers $n$ by mathematical induction. ✓

b) i)

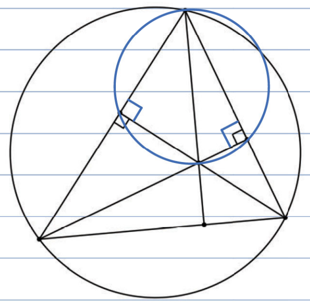


$$\angle CEB = 180° - \angle AEH \quad (\angle s \text{ on a straight line})$$
$$= 90°$$
$$= \angle CFB \quad (\text{given})$$

$\therefore$ CEFB is a cyclic quad.

(interval BC subtends equal $\angle$s on the same side so CEFB are concyclic)

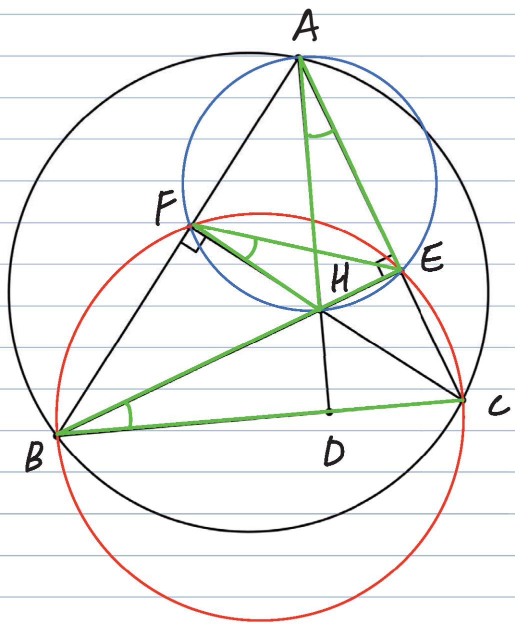
ii)



$$\angle AFH = 180° - \angle BFC \quad (\angle s \text{ on a straight line})$$
$$= 90°$$
$$= \angle AEH \quad (\text{given})$$

$\therefore$ EAFH is a cyclic quadrilateral

(opposite $\angle$s of a cyclic quad. are supplementary)




$\angle CAD = \angle EFH$ ($\angle$s in the same segment)

and

$\angle EFH = \angle EBC$ (similarly)

$\therefore \angle CAD = \angle EFH = \angle EBC$

iii)
$$\angle ADC = 180° - \angle CAD - \angle DCA \quad (\angle \text{ sum of } \Delta ADC)$$
$$= 180° - \angle CAD - (90° - \angle EBC)$$
$$= 90° - \angle CAD + \angle EBC \quad (\angle CAD = \angle EBC \text{ proved in part ii})$$
$$= 90°$$

$\therefore AD \perp BC$

c) i) $m_{PQ} = \dfrac{ap^2 - aq^2}{2ap - 2aq}$

$= \dfrac{(p+q)(p-q)}{2(p-q)}$

$= \dfrac{p+q}{2}$

chord PQ : $y - \cancel{ap^2} = \dfrac{p+q}{2}(x - 2ap)$

$= \dfrac{p+q}{2}x - ap(\cancel{p}+q)$

$y = \dfrac{p+q}{2}x - apq$

$(0,a) \to$ chord PQ : $a = 0 - apq$

$pq = -1$

Alternatively,

$m_{PS} = \dfrac{ap^2 - a}{2ap - 0}$ $\qquad m_{SQ} = \dfrac{aq^2 - a}{2aq - 0}$

$= \dfrac{p^2 - 1}{2p}$ $\qquad = \dfrac{q^2 - 1}{2q}$

$m_{PS} = m_{SQ}$

$\dfrac{p^2 - 1}{2p} = \dfrac{q^2 - 1}{2q}$

$q(p^2 - 1) = p(q^2 - 1)$

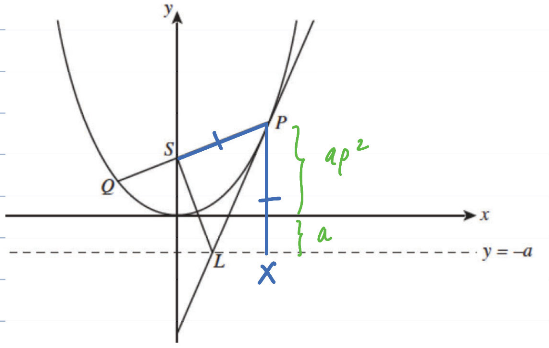
$p - q = pq^2 - qp^2$

$= pq(q - p)$

$pq = -1$

or some other valid method

ii)




$PS = PX$ (locus definition of parabola)

$= ap^2 + a$

$= a(p^2+1)$

iii) tangent p : $y = px - ap^2$

directrix : $y = -a$

$-a = px - ap^2$

$px = ap^2 - a$

$x = ap - \frac{a}{p}$

$\therefore L\left(ap - \frac{a}{p}, -a\right)$ ✓

$LS^2 = (2a)^2 + \left(ap - \frac{a}{p}\right)^2$

$= a^2\left(4 + \left(p - \frac{1}{p}\right)^2\right)$ ✓

$= a^2\left(4 + p^2 - 2 + \frac{1}{p^2}\right)$

$= a^2\left(p + \frac{1}{p}\right)^2$

$PS \times QS = a(p^2+1) \cdot a(q^2+1)$ ✓

$= a^2(p^2+1)(q^2+1)$

$= a^2(p^2+1)\left(\left(-\frac{1}{p}\right)^2 + 1\right)$

$= a^2(p^2+1)\left(\frac{1}{p^2} + 1\right)$

$= a^2\left(1 + p^2 + \frac{1}{p^2} + 1\right)$

$= a^2\left(p + \frac{1}{p}\right)^2$ ✓

$= LS^2$